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DESIGN OF FIBROUS COMPOSITES WITH ASSIGNED STRAIN–STRENGTH CHARACTERISTICS

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A solution of the problem of designing composites reinforced by high-modulus fibers with preassigned strain-strength characteristics is presented. The problem belongs to the class of ill-posed problems [1]. The problem is studied numerically; a numerical algorithm for solving the problem is proposed and some examples are considered.

1. Statement of the Problem of Calculation of Fibrous Composites. For calculating the averaged elastic constants $\{a_{ijkl}\}$ in the reinforcement plane of the composite (i, j, k, $l = 1$, 2 and the layers are taken parallel to the plane Ox_1x_2) and local stresses $\{\sigma^{\varepsilon}_{ij}\}$ in fibrous composites reinforced by periodically alternating layers of fibers (Young's modulus E of the fibers is much greater than that of the binder) the following formulas are obtained in [2, 3] (to the accuracy cited in [2] only axial stresses σ^{α}_{n} are nonzero in the fibers):

$$a_{ijkl} = ES \sum_{\alpha=1}^{M} \gamma_i^{\alpha}\gamma_j^{\alpha}\gamma_k^{\alpha}\gamma_l^{\alpha}\mu_{\alpha}; \tag{1.1}$$

in the αth layer of reinforcing fibers

$$\sigma^{\varepsilon}_{ij} = E\gamma_i^{\alpha}\gamma_j^{\alpha} \sum_{k,l=1,2} \gamma_k^{\alpha}\gamma_l^{\alpha} e_{kl}, \quad \sigma^{\alpha}_{n} = E \sum_{k,l=1,2} \gamma_k^{\alpha}\gamma_l^{\alpha} e_{kl}. \tag{1.2}$$

Here S is the volume content of fibers in the composite; μ_α is the specific (related to S) content of fibers in the αth reinforcing family; $\{\gamma_i^{\alpha}\}$ is the direction cosines of the αth family fiber axis; M is the number of reinforcing families over the period of the composite structure; $\{e_{kl}\}$ are the averaged strains of the composite [i.e., the strains determined by solving the problem of deformation of a material with elastic constants $\{a_{ijkl}\}$ (1.1)]. Note that formulas (1.1), (1.2) were used systematically before they were rigorously proved mathematically.

For the specific contents $\{\mu_\alpha\}$ the following relations hold:

$$\mu_\alpha \geqslant 0 \qquad (\alpha = 1, \ldots, M), \quad \sum_{\alpha=1}^{M} \mu_\alpha = 1. \tag{1.3}$$

When the fibers are arranged parallel to the plane Ox_1x_2 (Fig. 1), the direction cosines are $\gamma_1^{\alpha} = \cos\varphi_\alpha$, $\gamma_2^{\alpha} = \sin\varphi_\alpha$, $\gamma_3^{\alpha} = 0$, where φ_α is the angle between the fiber axes of the αth family and the Ox_1 axis.

2. Formulation of the Design Problem. If the composition and the structure of the composite are known, one can determine its averaged elastic characteristics and local stresses in the fibers using formulas (1.1) and (1.2). If the strength criterion of the fibers is known, for example, if it is taken as

$$0 \leqslant f(\sigma^{\varepsilon}_{ij}) \leqslant \sigma^*, \tag{2.1}$$

one can also judge the presence or absence of failure of fibers in the composite when averaged stresses $\{\sigma_{ij}\}$ are applied to it (the latter are determined from the averaged Hooke's law $\sigma_{ij} = a_{ijkl}e_{kl}$). The problem described is the problem of calculation of composites. Much attention has been paid to this problem in the works of different authors (see, for example, references in [2]).

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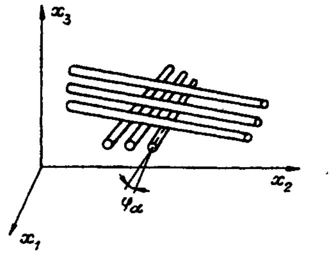

Fig. 1

Let us consider the design problem (DP) of a controlled-property composite, which is the reverse of the problem of calculation. Its descriptive formulation is as follows: what composition and structure a composite should have to possess an assigned set of averaged elastic characteristics $\{a_{ijkl}\}$ and withstand application of averaged stresses $\{\sigma_{ij}\}$ without fracture.

Remark 1. The composition and structure of this type of composites are described by the set of quantities S, M, $\{\varphi_\alpha\}$, and $\{\mu_\alpha\}$.

The design problem is ill-posed [4]. Consequently, its study without adequate mathematical methods provides little information. Let us consider the design problem from a mathematical viewpoint.

A Composite Possesses a Preassigned Set of Averaged Elastic Characteristics $\{a_{ijkl}\}$. As follows from (1.1), all the characteristics are expressed in terms of four functions of the arguments $\{\varphi_\alpha\}$ and $\{\mu_\alpha\}$:

$$\begin{aligned} y_1 &= \sum_{\alpha=1}^{M} \mu_\alpha \cos^4\varphi_\alpha, & y_2 &= \sum_{\alpha=1}^{M} \mu_\alpha \sin^4\varphi_\alpha, \\ y_3 &= \sum_{\alpha=1}^{M} \mu_\alpha \sin\varphi_\alpha \cos^3\varphi_\alpha, & y_4 &= \sum_{\alpha=1}^{M} \mu_\alpha \sin^3\varphi_\alpha \cos\varphi_\alpha. \end{aligned} \tag{2.2}$$

Solving (1.1) with respect to y_1, y_2, y_3, and y_4, we obtain the problem with respect to the unknowns S, M, $\{\varphi_\alpha\}$, and $\{\mu_\alpha\}$:

$$\begin{aligned} \sum_{\alpha=1}^{M} \mu_\alpha \cos^4\varphi_\alpha = y_1 = \frac{a_{1111}}{ES}, \qquad & \sum_{\alpha=1}^{M} \mu_\alpha \sin^4\varphi_\alpha = y_2 = \frac{a_{2222}}{ES}, \\ \sum_{\alpha=1}^{M} \mu_\alpha \sin\varphi_\alpha \cos^3\varphi_\alpha = y_3 = \frac{a_{1112}}{ES}, \qquad & \sum_{\alpha=1}^{M} \mu_\alpha \sin^3\varphi_\alpha \cos\varphi_\alpha = y_4 = \frac{a_{2221}}{ES}. \end{aligned} \tag{2.3}$$

This yields the relation $a_{1122} = a_{1212} = (1/2)(ES - a_{1111} - a_{2222})$ between the averaged elastic characteristics, which is the solvability condition for the system (1.1) with respect to y_1, y_2, y_3, and y_4.

Equation (2.3) and condition (1.3), which together form the convex combination problem (CCP) [1] relative to volume contents of the fibers $\{\mu_\alpha\}$, is a mathematical formulation of the condition that a composite possesses an assigned set of averaged characteristics $\{a_{ijkl}\}$. In the general case the solvability condition of the resulting problem is as follows.

Proposition 1 [5]. Problem (1.3), (2.3) at $M \geqslant 5$ has a solution if and only if a point $\mathbf{y} = (y_1, y_2, y_3, y_4)$ belongs to the set $\operatorname{conv}\Gamma$, where $\Gamma = \{(\cos^4\varphi,\ \sin^4\varphi,\ \sin\varphi\cos^3\varphi,\ \sin^3\varphi\cos\varphi) : \varphi \in \Phi\}$, Φ is a set of permissible laying angles, and conv is a convex hull.

Composites with Symmetrically Laid Fibers. Symmetrical laying of fibers characterized by the relations $\varphi_i = \varphi_{M-i}$, $\mu_i = \mu_{M-i}$ (M is an even integer) is frequently used in practice. In this case the last two relations from (2.2) vanish identically, and only two first equations remain in (2.3). The solvability condition of the problem (1.3), (2.3) follows from Proposition 1 if we set in it $\Gamma = \{(\cos^4\varphi, \sin^4\varphi) : \varphi \in \Phi\} = \{(\eta, (1-\sqrt{\eta})^2),\ \eta \in \cos^4\Phi\}$, where $\eta = \cos^4\varphi$, $\cos^4\Phi = \{\eta = \cos^4\varphi : \varphi \in \Phi\}$ and $M \geqslant 5$.

Fibers Remain Intact upon Application of the Averaged Stresses $\{\sigma_{ij}\}$ *to the Composite.* If $\{a_{ijkl}\}$ are specified, the averaged strains are given by the formula $e_{ij} = \{a_{ijkl}\}^{-1}\sigma_{kl}$. Substituting this expression into

(1.2), we obtain an averaged strength criterion below (which is called so because, in distinction to (2.1), it is formulated in terms of averaged stresses [6–8])

$$f_\alpha = f\Big(E\gamma_i^\alpha\gamma_j^\alpha \sum_{k,l=1,2} \gamma_k^\alpha\gamma_l^\alpha \sum_{m,n=1,2} \{a_{klmn}\}^{-1}\sigma_{mn}\Big) \leqslant \sigma^* \tag{2.4}$$

in the αth family of reinforcing fibers.

Since $\{\gamma_i^\alpha\}$ are expressed in terms of φ_α and $\{a_{ijkl}\}$ in terms of $\mathbf{y}$, one can rewrite (2.4) as

$$F(\varphi_\alpha, \mathbf{y}, \sigma_{mn}) \leqslant \sigma^* \tag{2.5}$$

in the αth family of reinforcing fibers. Here F is a known function [obtained from the left-hand side of (2.4) by replacement of $\{\gamma_i^\alpha\}$ and $\{a_{ijkl}\}$ by their expressions in terms of φ_α and $\mathbf{y}$]. To formulate the requirement of fulfillment of the strength criteria for all families of reinforcing fibers, we introduce the function

$$M(\sigma_{mn}, \{\varphi_\alpha\}) = \max F_\alpha(\varphi_\alpha, \mathbf{y}, \sigma_{mn})$$

(the maximum is taken over all laying angles φ_α for all families of fibers actually involved in the composite, i.e., of those fibers for which the condition $\mu_\alpha > 0$ holds) and require the fulfillment of the condition

$$M(\sigma_{mn}, \{\varphi_\alpha\}) \leqslant \sigma^*. \tag{2.6}$$

Inequality (2.6) is a mathematical formulation of the condition of the absence of fracture for composite fibers under prescribed averaged stresses.

Remark 2. Only the laying angle of fibers of the αth family φ_α appears as an argument in the strength condition of this family. This is used below.

Typical Design Problems (Mathematical Formulations).

1) Designing a composite with a preassigned set of averaged characteristics $\{a_{ijkl}\}$: one should solve the problem (1.3), (2.3).

2) Designing a composite of maximum strength with a preassigned set of averaged elastic characteristics $\{a_{ijkl}\}$: it is required to solve the problem (1.3), (2.3); in addition, the left-hand side of (2.6) should contain a minimum value $M(\sigma_{mn}, \{\varphi_\alpha\}) \to \min$. $1/M(\sigma_{mn}, \{\varphi_\alpha\})$ has the meaning of the strength reserve.

3) Designing a composite with a preassigned set of averaged elastic characteristics, which remains intact after application of preassigned averaged stresses: it is required to solve the problem (1.3), (2.3), (2.6); (2.6) in this case should be valid for averaged stresses σ_{ij} from a certain preassigned set Σ.

4) Designing a composite with a preassigned set of averaged characteristics $\{a_{ijkl}\}$ with minimum volume of fibers: it is required to solve the problem (1.3), (2.3); in addition, the quantity S (the volume content of fibers) should be given a minimum value $S \to \min$. Note that if the fiber is heavier than the binder (which is the case, as a rule), the problem stated is also the problem of designing a composite of minimum weight.

Discrete Design Problem. It is often inexpedient or difficult to consider the problem with infinite number of possible laying angles. In this connection, a discrete design problem which appears in the case where the set of possible laying angles has the form $\Phi = \{\varphi_\beta,\ \beta = 1, \ldots, N\}$ is of considerable interest. At large N the discrete problem approximates the continuous one (for details see [9]).

Method of Numerical Solution of the Design Problem. Let us introduce the vectors $\mathbf{y}_\beta = (\cos^4\varphi_\beta, \sin^4\varphi_\beta, \sin\varphi_\beta\cos^3\varphi_\beta, \sin^3\varphi_\beta\cos\varphi_\beta) \in R^4$, $\beta = 1, \ldots, N$. Then problem (1.3), (2.2) can be written in the form

$$\begin{gathered}\sum_{\alpha=1}^{M} y_{1\alpha}\mu_\alpha = y_1; \quad \mu_\alpha \geqslant 0; \quad \alpha = 1, \ldots, M; \quad \sum_{\alpha=1}^{m} \mu_\alpha = 1, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \sum_{\alpha=1}^{M} y_{4\alpha}\mu_\alpha = y_4. \end{gathered} \tag{2.7}$$

Let us solve the resulting problem by the method of convolution of systems of linear equations [10] (called so by analogy with the method of convolution of systems of linear inequalities [11]). The method is based on the solvability of a one-dimensional CCP

$$\sum_{\beta=1}^{m} y_\beta \mu_\beta = x, \ \mu_\beta \geqslant 0 \quad (\beta = 1,\dots,m), \quad \sum_{\beta=1}^{m} \mu_\beta = 1 \tag{2.8}$$

in explicit form.

Proposition 2.

1. Problem (2.8) is solvable if and only if the *condition P*: $y_1 \leqslant x \leqslant y_m$ holds [without loss of generality it is assumed that points are ordered in increasing order in (2.8): $y_1 < y_2 < \dots < y_m$].

2. Let the *condition P* be satisfied, then the point x is representable in the form $x = \lambda_a y_a + \lambda_b y_b$ as a convex combination of the points $\{y_a\}$ and $\{y_b\}$ such that $y_a \leqslant x$ and $y_b > x$. For the given a and b, we introduce the notation

$$S_\eta = \{S_{\eta\beta}\} = (0,\dots,0,\ \underset{\text{at the } a\text{th place}}{\lambda_a},\ 0,\dots,0,\ \underset{\text{at the } b\text{th place}}{\lambda_b},\ 0,\dots,0) \in R^m. \tag{2.9}$$

Then the set of solutions of the problem (2.9) is defined by the convolution formula

$$\mu_\beta = \sum_{\eta=1}^{M_1} S_{\eta\beta}\lambda_\eta. \tag{2.10}$$

Here M_1 is the total number of segments of the form $[y_a,\ y_b]$ containing the point x and $\{\lambda_\eta\}$ are arbitrary numbers satisfying the condition

$$\lambda_\eta \geqslant 0 \quad (\eta = 1,\dots,M_1), \qquad \sum_{\eta=1}^{M_1} \lambda_\eta = 1. \tag{2.11}$$

Remark 3. The values of λ_a and λ_b from (2.9) are expressed as $\lambda_a = (x - y_a)/(y_b - y_a)$ and $\lambda_b = 1 - \lambda_a$.

Convolution of the system (2.7) is carried out as follows. The first equation in (2.7) is a one-dimensional CCP. Its solution, if the problem is solvable (see item 1 of Proposition 2), has the form (2.10), (2.11). We substitute (2.10) into the second, third, and fourth equations (2.7). Changing the summation order, we obtain

$$\sum_{\eta_1=1}^{M_1} \lambda_{\eta_1}\Big(\sum_{\alpha=1}^{M} y_{2\alpha} S_{\eta_1\alpha}\Big) = y_2,$$

$$\dots\dots\dots\dots\dots\dots \tag{2.12}$$

$$\sum_{\eta_1=1}^{M_1} \lambda_{\eta_1}\Big(\sum_{\alpha=1}^{M} y_{4\alpha} S_{\eta_1\alpha}\Big) = y_4.$$

Denote

$$y_{\eta_1} = \sum_{\alpha=1}^{M} y_\alpha S_{\eta_1\alpha}.$$

As is evident, the first equation in (2.12) together with (2.11) is again a one-dimensional CCP whose solution is found on the basis of Proposition 1. As a result, we solve all four equations in (1.3), (2.2) in four steps of the algorithm proposed (with the proviso that in each step the *condition P* is satisfied). The solution of the problem (1.3), (2.2) (if any) is given by the formula

$$\mu_\alpha = \sum_{\eta_4=1}^{M_4} \lambda_{\eta_4} P_{\eta_4\alpha}, \quad \lambda_{\eta_4} \geqslant 0, \quad \sum_{\eta_4=1}^{M_4} \lambda_{\eta_4} = 1, \tag{2.13}$$

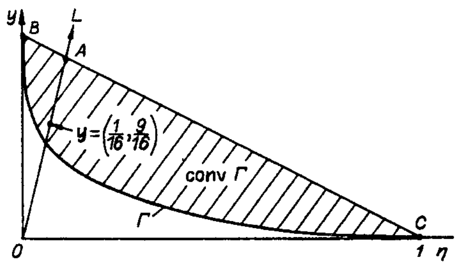

Fig. 2

where

$$\mathbf{P}_{\eta_4} = \{P_{\eta_4\alpha}\} = \left\{ \sum_{\eta_1=1}^{M_1} \sum_{\eta_2=1}^{M_2} \sum_{\eta_3=1}^{M_3} S_{\eta_4\eta_3} S_{\eta_3\eta_2} S_{\eta_2\eta_1} S_{\eta_1\alpha} \right\}_{\alpha=1}^{M}.$$

As follows from (2.13), the number of solutions of problem (1.3), (2.2) is infinite and is expressed in terms of the finite number of vectors $\{\mathbf{P}_{\eta_4}\}$.

Remark 4. After replacement of M_4 by M_k, the formula (2.13) gives the solution of the first k equations. As was shown in [8], the set of solutions given by (2.13) remains unchanged if vectors having more than $k+1$ nonzero coordinates are excluded during convolution in the kth step (by analogy with [11] this method is called reduced convolution).

Remark on the Methods of Solving the Design Problem. The problem under consideration is reduced to a CCP, which can be solved graphically for two equations (the CCP that appears for symmetrical laying), and only numerically in the case of greater dimensionality. Let us cite examples of solutions for both cases.

Example 1. Let it be required to create a composite with a symmetrical structure and the averaged elastic characteristics $a_{1111} = 0.15 \cdot 10^{11}$ Pa and $a_{2222} = 0.03 \cdot 10^{11}$ Pa from a fiber with Young's modulus $E = 0.7 \cdot 10^{11}$ Pa (fiberglass) with the given volume content of fibers $S = 0.4$ and minimum volume content of fibers.

From (2.2), we obtain (by virtue of symmetry only two first equations are considered)

$$\mathbf{y} = (y_1, y_2) = \left(\frac{1}{16}, \frac{9}{16}\right) \qquad \text{at} \quad S = 0.4.$$

The set $\Gamma = \{\eta, (1-\sqrt{\eta})^2 : \eta \in [0,1]\}$ is shown in Fig. 2, from which it is evident see that $\mathbf{y} \in \operatorname{conv}\Gamma$ and the problem is solvable. An infinite number of designs are possible. Let us find a design with minimum content of fibers (i.e., satisfying the condition $S \to \min$). Let S vary from 0 to 1. In this case the point $\mathbf{y}$ moves along the ray L, as shown in Fig. 2. The lowest value of S at which $\mathbf{y}$ still belongs to $\operatorname{conv}\Gamma$ corresponds to the point A and equals 0.25. The appropriate design of a composite with minimum fiber content is $\mu_1 = BA/BC = 0.9$, $\mu_2 = AC/BC = 0.1$ and the laying angles are $\varphi_1 = 90°$, $\varphi_2 = 0$.

Example 2. Let it be required to create a composite with the following averaged elastic characteristics: $a_{1111} = 0.25 \cdot 10^{11}$ Pa, $a_{2222} = 0.1 \cdot 10^{11}$ Pa. The fibers with Young's modulus $E = 0.7 \cdot 10^{11}$ Pa are used (fiberglass). The volume content of fibers is $S = 0.6$.

Let there be no restrictions on the laying angles $\Phi = [-\pi/2, \pi/2]$. After discretization of the interval $[-\pi/2, \pi/2]$ with the step $\delta = \pi/15$ we obtain a computer-solvable discrete DP. From calculations (on a ES 1033 computer; the computation time is $\sim$ 1 min, including translation), we obtained $M_4 = 281$ vectors $\{\mathbf{P}_{\eta_4}\}$. We cite some of them:

$$\mathbf{P}_1 = (0.1736,\ 0.0545,\ 0,\ 0,\ 0,\ 0,\ 0.4009,\ 0,\ 0,\ 0.2285,\ 0.1426,\ 0,\ 0,\ 0,\ 0), \ldots,$$

$$\mathbf{P}_{107} = (0,\ 0,\ 0.2116,\ 0,\ 0,\ 0,\ 0,\ 0.3946,\ 0.1819,\ 0,\ 0.0004,\ 0,\ 0.2115,\ 0,\ 0), \ldots .$$

The set of designs of a composite with the averaged characteristics preassigned above has the form

$$\mu_1 = 0.1736\lambda_1 + \cdots + 0 \cdot \lambda_{107} + \cdots, \qquad \mu_9 = 0 \cdot \lambda_1 + \cdots + 0.1819\lambda_{107} + \cdots,$$
$$\mu_2 = 0.0545\lambda_1 + \cdots + 0 \cdot \lambda_{107} + \cdots, \qquad \mu_{10} = 0.2285\lambda_1 + \cdots + 0 \cdot \lambda_{107} + \cdots,$$
$$\mu_3 = 0 \cdot \lambda_1 + \cdots + 0.2116\lambda_{107} + \cdots, \qquad \mu_{11} = 0.1426\lambda_1 + \cdots + 0.004\lambda_{107} + \cdots,$$
$$\mu_4 = 0 \cdot \lambda_1 + \cdots + 0 \cdot \lambda_{107} + \cdots, \qquad \mu_{12} = 0 \cdot \lambda_1 + \cdots + 0 \cdot \lambda_{107} + \cdots,$$
$$\mu_5 = 0 \cdot \lambda_1 + \cdots + 0 \cdot \lambda_{107} + \cdots, \qquad \mu_{13} = 0 \cdot \lambda_1 + \cdots + 0.2115\lambda_{107} + \cdots,$$
$$\mu_6 = 0 \cdot \lambda_1 + \cdots + 0 \cdot \lambda_{107} + \cdots, \qquad \mu_{14} = 0 \cdot \lambda_1 + \cdots + 0 \cdot \lambda_{107} + \cdots,$$
$$\mu_7 = 0.4009\lambda_1 + \cdots + 0 \cdot \lambda_{107} + \cdots, \qquad \mu_{15} = 0 \cdot \lambda_1 + \cdots + 0 \cdot \lambda_{107} + \cdots,$$
$$\mu_8 = 0 \cdot \lambda_1 + \cdots + 0.3946\lambda_{107} + \cdots,$$

where $\lambda_1, \ldots, \lambda_{107}, \ldots, \lambda_{281} \geqslant 0;\ \lambda_1 + \cdots + \lambda_{107} + \cdots + \lambda_{281} = 1.$

3. Designing a Maximum-Strength Composite. As was noted in Remark 2, the strength of the fibers of the αth family is determined by their laying angle. In view of this, the problem $M(\sigma_{mn}, \{\varphi_\alpha\}) \to \min$ subject to conditions (1.3), (2.3) will be solved if we are able to select, among the possible laying angles of fibers $\{\varphi_\beta,\ \beta = 1, \ldots, N\}$, the families $\{\varphi_\alpha\}$ satisfying the two conditions:

1) the conditions (1.3), (2.3) are fulfilled for the family $\{\varphi_\alpha\}$,

2) the quantity $\max_\alpha\{f_\alpha\}$ is minimum for the family $\{\varphi_\alpha\}$.

To satisfy these conditions we proceed as follows. We enumerate the laying angles of fibers in increasing order of the values $\{f_\beta\}$ [determined in (2.4)]. We take the angle φ_1 and check whether relations (1.3) and (2.3) hold for the vector $\mathbf{y}_1$ corresponding to the angle. If not, we add the angle φ_2 to φ_1 and so on, until for a certain family $\{\varphi_1, \ldots, \varphi_K\}$ ($\{\mathbf{y}_1, \ldots, \mathbf{y}_K\}$) relations (1.3), (2.3) hold for the first time. The fulfillment of relations (1.3), (2.3) is checked by the above-mentioned methods. After that, the solution of the problem $M(\sigma_{mn}, \{\varphi_\alpha\}) \to \min$ subject to conditions (1.3), (2.3) is given by the formula (2.13), but the vectors $\{\mathbf{y}_1, \ldots, \mathbf{y}_K\}$ should be used instead of the vectors $\{\mathbf{y}_1, \ldots, \mathbf{y}_M\}$. In this case the equality below is valid:

$$\min M(\sigma_{mn},\ \{\varphi_\alpha\}) = f_K. \tag{3.1}$$

Remark 5. The DP of the highest-strength composite is solvable if and only if the DP of a composite with preassigned averaged characteristics is solvable. In view of (3.1) the strength condition for the highest-strength composite can be written as $f_K \leqslant \sigma^*$ (at $f_K > \sigma^*$ some families of reinforcing fibers fail).

Remark on the Methods of Solving the Problem. The problem (appearing in the case of symmetrical laying) can be solved graphically for two equations and numerically in the case of greater dimensionality.

Example 3. It is required to create a composite with averaged elastic characteristics $a_{1111} = 0.15 \cdot 10^{11}$ Pa and $a_{2222} = 0.03 \cdot 10^{11}$ Pa. Fibers with Young's modulus $E = 0.7 \cdot 10^{11}$ Pa (iberglass) are used. The given volume content of the fibers is $S = 0.4$. The composite should have maximum possible strength upon application of an averaged stress of the type $\sigma_{11} \neq 0$, $\sigma_{ij} = 0$ at $ij \neq 11$ (tension along the Ox_1 axis).

We us take the strength condition of the fiber material as

$$f(\sigma^\varepsilon_{ij}) = |\sigma^\alpha_n| \leqslant \sigma^* \quad \text{(for fiberglass} \quad \sigma^* = 0.024 \cdot 10^{11} \text{ Pa).} \tag{3.2}$$

From (1.2), the stresses in fibers of the αth family are

$$\sigma^\alpha_n = E \sum_{k,l=1,2} \gamma^\alpha_k \gamma^\alpha_l e_{kl}.$$

The averaged strains in this case have the form

$$e_{11} = \frac{a_{1111}\sigma_{11}}{\Delta}, \qquad e_{22} = -\frac{a_{1122}\sigma_{11}}{\Delta}, \qquad e_{12} = 0,$$

where $\Delta = a_{1111}a_{2222} - a^2_{1122}$. The value a_{1122} is calculated from a_{1111} and a_{2222} (2.2) and equals $0.05 \cdot 10^{11}$ Pa. Hence,

$$\sigma^\alpha_n = E(a_{1111}\cos^2\varphi_\alpha - a_{1122}\sin^2\varphi_\alpha)\sigma_{11}/\Delta.$$

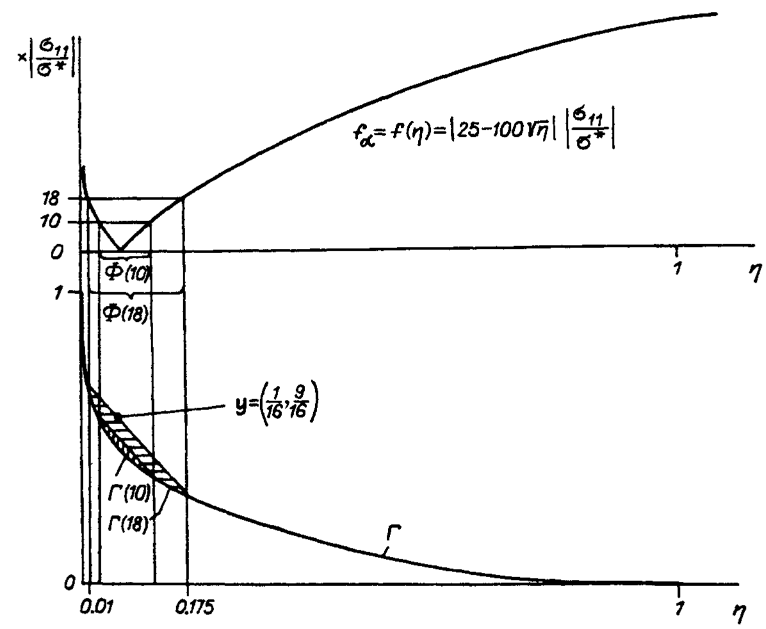

Fig. 3

Substituting this expression in the local strength condition, we obtain the averaged strength criterion

$$\sigma_n^\alpha = E(a_{1111}\cos^2\varphi_\alpha - a_{1122}\sin^2\varphi_\alpha)\sigma_{11}/\Delta \leqslant 1 \tag{3.3}$$

in the αth family of reinforcing fibers.

Let us consider the plot of the function $f_\alpha = f(\eta) = |(25 - 100\sqrt{\eta})\sigma_{11}/\sigma^*|$ obtained from (3.3) at $\eta = \cos^4\varphi$ (shown in the upper part of Fig. 3). In the lower part of Fig. 3 the line $\Gamma = \{(\eta, (1-\sqrt{\eta})^2) : \eta \in [0,1]\}$ is drawn and the system of increasing sets $\Phi(\sigma) = \{f(\eta) \leqslant \sigma\}$ and $\Gamma(\sigma) = \{(\eta, (1-\sqrt{\eta})^2) : \eta \in \Phi(\sigma)\}$, and also the point $\mathbf{y} = (1/16, 9/16)$, which is a solution of the first two equations (2.3) (see remark on symmetrical laying), are shown. The point $\mathbf{y}$ is first covered by the set $\operatorname{conv}\Gamma(\sigma)$ at $\sigma = \sigma_{\min} = 18|\sigma_{11}/\sigma^*|$. Accordingly, the design for the highest-strength composite is as follows: the fiber laying angles $\varphi_{1.4} = \pm 72^\circ$ and $\varphi_{2,3} = \pm 50^\circ$ and the content of fibers in reinforcing families $\mu_{1.4} = 0.16$ and $\mu_{2,3} = 0.34$. The strength criterion for the material designed is representable in the form $\sigma_{\min} \leqslant 1$ or $18|\sigma_{11}| \leqslant \sigma^*$.

Example 4. It is required to design a composite with a maximum strength with the same averaged elastic characteristics as in Example 2. Let the averaged stresses $\sigma_{11} = 0.01 \cdot 10^{11}$ Pa and $\sigma_{ij} = 0$ if $ij \neq 11$ (axial tension along the Ox_1 axis) be applied to the composite. The strength conditions for the fiber material are taken in the form (3.2). The averaged strength criterion has the form (3.3).

We calculate the values $\{f_\beta\}$ and renumber $\{\varphi_\beta\}$ in increasing order of $\{f_\beta\}$ (see Table 1, where the values β correspond to the initial numeration, while γ to the new one). After that, we solve the problem (1.3), (2.2) for the families $\Phi_k = \{\varphi_1, \ldots, \varphi_k\}$ beginning from $k = 1$ until at a certain $k = K$ it turns out to be solvable for the first time. It was found in numerical calculations (EC 1033, calculation time $\sim$ 10 min) that $K = 11$, $f_K = f_{11} = 0.9529$, and the set of solutions of the problem (1.3), (2.2) for the system of vectors $\{\mathbf{y}_1, \ldots, \mathbf{y}_{11}\}$ has the form

$$M_4 = 7,$$

$$\mathbf{P}_1 = (0, 0, 0.2116, 0, 0, 0, 0, 0.3948, 0.1813, 0.0007, 0, 0, 0.2115, 0, 0),$$

$$\mathbf{P}_2 = (0, 0, 0.2116, 0, 0, 0, 0, 0.3946, 0.1819, 0.0007, 0, 0, 0.2115, 0, 0),$$

$$\mathbf{P}_3 = (0, 0, 0.2116, 0, 0, 0, 0, 0.3945, 0.1821, 0, 0, 0, 0.2113, 0, 0),$$

TABLE 5

γ	β	f_β
1	6	0.0154
2	11	0.0154
3	7	0.2713
4	10	0.2713
5	5	0.3121
6	12	0.3121
7	8	0.4113
8	9	0.4113
9	4	0.6546
10	13	0.6546
11	3	0.9528
12	14	0.9528
13	2	1.1552
14	15	1.1552
15	1	1.2268

$$\mathbf{P}_4 = (0, 0, 0.2114, 0, 0, 0.0003, 0, 0.3938, 0.1828, 0, 0, 0, 0.2116, 0, 0),$$

$$\mathbf{P}_5 = (0, 0, 0.2115, 0, 0, 0.0003, 0, 0.3938, 0.1828, 0, 0, 0, 0.2116, 0, 0),$$

$$\mathbf{P}_6 = (0, 0, 0.2114, 0, 0, 0.0003, 0, 0.3938, 0.1828, 0, 0, 0, 0.2116, 0, 0),$$

$$\mathbf{P}_7 = (0, 0, 0.2116, 0, 0, 0.0006, 0, 0.3932, 0.1830, 0, 0, 0, 0.2116, 0, 0).$$

It is evident that the solutions obtained are close to one another (coincidence of some of them is the result of computer approximation). An explanation to this can be found in [7]. By virtue of the foregoing, any solution obtained, for example $\mathbf{P}_2$ (coincident with the solution $\mathbf{P}_{107}$ from Example 2), can be taken as the final solution. The corresponding design of the maximum strength composite is as follows: the specific content of fibers in reinforcing families is

family $\mu_3 = 0.2116$, laying angle $\varphi_3 = -\pi/2 + 3\pi/15$,
family $\mu_8 = 0.3946$, laying angle $\varphi_8 = -\pi/2 + 8\pi/15$,
family $\mu_9 = 0.1819$, laying angle $\varphi_9 = -\pi/2 + 9\pi/15$,
family $\mu_{11} = 0.0004$, laying angle $\varphi_{11} = -\pi/2 + 11\pi/15$,
family $\mu_{13} = 0.2115$, laying angle $\varphi_{13} = -\pi/2 + 13\pi/15$.

The remaining reinforcing families are not used.

The strength condition of the composite designed (see Remark 6) can be written as $f_{11} \leqslant 1$. Since for the resulting design $f_{11} = 0.9529$, the composite designed endures the load applied. It should be noted that the composite fabricated in accordance with the design $\mathbf{P}_1$ from Example 2 would fail at the same averaged load, namely: the fibers of the first ($f_1 = 1.2268 > 1$), second, and fifteenth ($f_2 = f_{15} = 1.1552 > 1$) reinforcing families would fail (see Table 1).

Remark 6. In Example 4, the strength criterion of fibers is the first-order criterion [i.e., $f(t\sigma_{11}) = |t| f(\sigma_{11})$]. This enables one to calculate the ultimate strength of the composite designed under uniaxial tension $-\sigma_1^*$ along the Ox_1 axis. We have $f_{11} = \sigma^* |\sigma_1^*/0.01| f(0.01) = 1$, where $\sigma_1^* = \sigma^* 0.01/f(0.01) = \sigma^* 1.045$ Pa.

Designing a Composite Withstanding Pressigned Averaged Loads. Let it be required to design a composite with a preassigned set of averaged characteristics withstanding averaged stresses $\sigma_{ij} \in \Sigma$ (i.e., stresses from a certain class Σ). In view of (2.4) and (2.5), this means that the following requirements should be satisfied:

1) Only those families of reinforcing fibers are involved in the composite for which

$$f_\alpha = F(\varphi_\alpha, \mathrm{y}, \sigma_{mn}) \leqslant \sigma^* \quad \text{for all} \quad \sigma_{mn} \in \Sigma; \tag{3.4}$$

2) The conditions (1.3), (2.2) are satisfied.

Thus, to solve the problem, it is sufficient to select laying angles satisfying (3.4) and then solve for them the problem (1.3), (2.2) (methods for solving the problem are described above).

Remark 7. Naturally, the question about allowance for the strength of the second component (binder) arises, which can be solved on the basis of further development of the methods proposed above (see [12, 13]). Note that qualitative criteria that enable detection of the least strong component of the composite are reported in [14], and the averaged criteria for the absence of failure of the binder of fibrous composites reinforced by high-modulus fibers are obtained in [15–20].

Remark 8. In the general case, DP and CCP have a great number of solutions. In this connection, it should be noted that the set given by the algorithm is, as a rule, optimal (i.e., cannot be reduced without loss of solutions). An attempt to reduce the set of solutions obtained is equivalent to transition to the search for particular solutions [21]. One of the ways of doing this transition is to formulate problems of optimal designing (understood in a narrow sense as the statement of problems containing a minimized/maximized function). The reduction of the set of solutions in such cases (up to uniqueness of the solution) can be observed in the above examples.

It should be noted that interest in DP in the statements presented in this work is constant, although progress in the research of this problem was hindered because of the application of inadequate mathematical methods. As an example, we refer to the monograph [22], in which an analog of CCP (1.3), (2.2) is formulated and discussed, and which demonstrates that without using adequate mathematical methods the CCP cannot be efficiently studied.

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